

# Asymptotic behaviour of principal eigenvalues for a class of cooperative systems <sup>☆</sup>

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## Abstract

This paper analyzes the asymptotic behaviour as  $\lambda \uparrow \infty$  of the principal eigenvalue of the cooperative operator

$$\mathfrak{L}(\lambda) := \begin{pmatrix} L_1 + \lambda a(x) & -b \\ -c & L_2 + \lambda d(x) \end{pmatrix}$$

in a bounded smooth domain  $\Omega$  of  $\mathbb{R}^N$ ,  $N \geq 1$ , under homogeneous Dirichlet boundary conditions on  $\partial\Omega$ , where  $a \geq 0$ ,  $d \geq 0$ , and  $b(x) > 0$ ,  $c(x) > 0$ , for all  $x \in \bar{\Omega}$ . Precisely, our main result establishes that if  $\text{Int}(a + d)^{-1}(0)$  consists of two components,  $\Omega_{0,1}$  and  $\Omega_{0,2}$ , then

$$\lim_{\lambda \uparrow \infty} \sigma_1[\mathfrak{L}(\lambda); \Omega] = \min_{i \in \{1,2\}} \sigma_1[\mathfrak{L}(0); \Omega_{0,i}],$$

where, for any  $D \subset \Omega$  and  $\lambda \in \mathbb{R}$ ,  $\sigma_1[\mathfrak{L}(\lambda); D]$  stands for the principal eigenvalue of  $\mathfrak{L}(\lambda)$  in  $D$ . Moreover, if we denote by  $(\varphi_\lambda, \psi_\lambda)$  the principal eigenfunction associated to  $\sigma[\mathfrak{L}(\lambda); \Omega]$ , normalized so that  $\int_\Omega (\varphi_\lambda^2 + \psi_\lambda^2) = 1$ , and, for instance,

$$\sigma_1[\mathfrak{L}(0); \Omega_{0,1}] < \sigma_1[\mathfrak{L}(0); \Omega_{0,2}],$$

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then the limit

$$(\Phi, \Psi) := \lim_{\lambda \uparrow \infty} (\varphi_\lambda, \psi_\lambda)$$

is well defined in  $H_0^1(\Omega) \times H_0^1(\Omega)$ ,  $\Phi = \Psi = 0$  in  $\Omega \setminus \Omega_{0,1}$  and  $(\Phi, \Psi)|_{\Omega_{0,1}}$  provides us with the principal eigenfunction of  $\sigma[\mathfrak{L}(0); \Omega_{0,1}]$ . This is a rather striking result, for as, according to it, the principal eigenfunction must approximate zero as  $\lambda \uparrow \infty$  if  $a + d > 0$ , in spite of the cooperative structure of the operator.

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## 1. Introduction

This paper ascertains the limiting behaviour as  $\lambda \uparrow \infty$  of the lowest real eigenvalue and normalized associated eigenfunction of the linear eigenvalue problem

$$\begin{cases} (L_1 + \lambda a)\varphi - b\psi = \tau\varphi & \text{in } \Omega, \\ (L_2 + \lambda d)\psi - c\varphi = \tau\psi & \text{in } \Omega, \\ (\varphi, \psi) = (0, 0) & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^N$ ,  $N \geq 1$ , with boundary  $\partial\Omega$  of class  $C^{2,\nu}$  for some  $\nu \in (0, 1)$ , and the following assumptions are satisfied:

(A1) For each  $k \in \{1, 2\}$ , the second order differential operator

$$L_k = - \sum_{i,j=1}^N \alpha_{[ij,k]} D_i D_j + \sum_{i=1}^N \alpha_{[i,k]} D_i + \alpha_{[0,k]} \quad (1.2)$$

is uniformly strongly elliptic in  $\bar{\Omega}$ , i.e., there exists  $\mu_k > 0$  such that

$$\sum_{i,j=1}^N \alpha_{[ij,k]}(x) \xi_i \xi_j \geq \mu_k |\xi|^2 \quad (1.3)$$

for all  $x \in \bar{\Omega}$  and  $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ . Also, for any  $i, j \in \{0, \dots, N\}$ ,

$$\alpha_{[ij,k]} \in C^{2,\nu}(\bar{\Omega}), \quad \alpha_{[i,k]} \in C^{1,\nu}(\bar{\Omega}), \quad \alpha_{[0,k]} \in C^v(\bar{\Omega}).$$

Without loss of generality, we can assume that

$$\alpha_{[ij,k]} = \alpha_{[ji,k]}, \quad k = 1, 2.$$

(A2)  $a, b, c, d \in C^v(\bar{\Omega})$  and (1.1) is *strongly cooperative* in the sense that

$$b(x) > 0 \quad \text{and} \quad c(x) > 0 \quad \text{for all } x \in \bar{\Omega}. \quad (1.4)$$

Moreover,  $a \geq 0, d \geq 0$ , the open set

$$\Omega_+ := \{x \in \Omega: a(x) + d(x) > 0\}$$

is a subdomain of  $\Omega$  of class  $C^{2,v}$  with  $\bar{\Omega}_+ \subset \Omega$ , and the compact set

$$K_0 := (a + d)^{-1}(0) = \bar{\Omega} \setminus \Omega_+$$

consists of two disjoint components,  $K_{0,i}, i \in \{1, 2\}$ , such that  $K_{0,2} \subset \Omega$  and  $\partial K_{0,i}$  are of class  $C^{2,v}$  for each  $i \in \{1, 2\}$ . Throughout this paper, we will set

$$\Omega_{0,i} := \text{Int } K_{0,i} \subset \Omega, \quad i \in \{1, 2\}.$$

(A3) Setting

$$\mathcal{S}(V_1, V_2) := \begin{pmatrix} L_1 + V_1 & -b \\ -c & L_2 + V_2 \end{pmatrix}, \quad V_1, V_2 \in C^v(\bar{\Omega}), \quad (1.5)$$

and  $\mathcal{S}_0 := \mathcal{S}(0, 0)$ , we assume the following estimate to be satisfied

$$\sigma_1[\mathcal{S}_0; \Omega_{0,1}] < \sigma_1[\mathcal{S}_0; \Omega_{0,2}], \quad (1.6)$$

where  $\sigma_1[\mathcal{S}_0; \Omega_{0,i}]$  stands for the principal eigenvalue of  $\mathcal{S}_0$  in  $\Omega_{0,i}, i \in \{1, 2\}$ .

Under condition (1.4), the differential operator (1.5), and, hence, the linear eigenvalue problem (1.1), is strongly cooperative, as discussed by de Figueiredo and Mitidieri [7], Sweers [15], López-Gómez and Molina-Meyer [12], and Amann [3]. Consequently, for any smooth subdomain  $D \subset \Omega$ , the principal eigenvalue  $\sigma_1[\mathcal{S}(V_1, V_2); D]$  is well defined. Section 4 will show that (1.6) holds if the Lebesgue measure of  $\Omega_{0,2}$ , denoted by  $|\Omega_{0,2}|$ , is sufficiently small. We recall that  $\sigma_1[\mathcal{S}(V_1, V_2); D]$  is the unique value of  $\tau$  for which

$$\begin{cases} (L_1 + V_1)\varphi - b\psi = \tau\varphi \\ (L_2 + V_2)\psi - c\varphi = \tau\psi \\ (\varphi, \psi) = (0, 0) \end{cases} \quad \begin{array}{l} \text{in } D, \\ \\ \text{on } \partial D, \end{array} \quad (1.7)$$

possesses a solution pair  $(\varphi, \psi)$  with  $\varphi > 0$  and  $\psi > 0$ , and that

$$\text{Re } \tau > \sigma_1[\mathcal{S}(V_1, V_2); D] \quad (1.8)$$

for any other eigenvalue  $\tau$  of (1.7).

Throughout this paper, for any given  $\lambda \in \mathbb{R}$ , we denote by  $\sigma(\lambda)$  the principal eigenvalue of (1.1), i.e.,

$$\sigma(\lambda) := \sigma_1[\mathcal{S}(\lambda a, \lambda d); \Omega], \quad \lambda \in \mathbb{R}, \quad (1.9)$$

and  $(\varphi_\lambda, \psi_\lambda)$  stands for the principal eigenfunction associated to  $\sigma(\lambda)$  such that

$$\int_{\Omega} (\varphi_\lambda^2 + \psi_\lambda^2) = 1. \quad (1.10)$$

It is well known that  $\varphi_\lambda \gg 0$  and  $\psi_\lambda \gg 0$ . In this paper, a function  $w \in \mathcal{C}^1(\bar{\Omega})$  is said to satisfy  $w \gg 0$  if  $w(x) > 0$  for all  $x \in \Omega$  and  $\partial w / \partial n(x) < 0$  for all  $x \in w^{-1}(0) \cap \partial\Omega$ , where  $n = n(x)$  stands for the outward unit normal to  $\partial\Omega$  at  $x \in \partial\Omega$ .

The main result of this paper can be stated as follows.

**Theorem 1.1.** *Suppose (A1)–(A3). Then,*

$$\lim_{\lambda \uparrow \infty} \sigma_1[\mathcal{S}(\lambda a, \lambda d); \Omega] = \sigma_1[\mathcal{S}_0; \Omega_{0,1}] \quad (1.11)$$

and

$$\lim_{\lambda \uparrow \infty} (\|\varphi_\lambda - \Phi_\omega\|_{H_0^1(\Omega)} + \|\psi_\lambda - \Psi_\omega\|_{H_0^1(\Omega)}) = 0, \quad (1.12)$$

where

$$\Phi_\omega := \Psi_\omega := 0 \quad \text{in } \Omega \setminus \Omega_{0,1}, \quad (1.13)$$

and

$$(\varphi_{0,1}, \psi_{0,1}) := (\Phi_\omega, \Psi_\omega)|_{\Omega_{0,1}}$$

provides us with the principal eigenfunction of  $\sigma_1[\mathcal{S}_0; \Omega_{0,1}]$  normalized by

$$\int_{\Omega_{0,1}} (\varphi_{0,1}^2 + \psi_{0,1}^2) = 1.$$

Theorem 1.1 is a substantial extension of some existing results for the very special case of the scalar equation (cf. López-Gómez [10,11], Dancer and López-Gómez [5], and [2]), and of a theorem of Molina-Meyer [13], where (1.11) was established for the very special case when

$$a^{-1}(0) = d^{-1}(0). \quad (1.14)$$

The reader is sent to these references and to [1] for further details about the applications of this type of results. Rather strikingly, in the absence of (1.14), Theorem 1.1 establishes that  $(\varphi_\lambda, \psi_\lambda)$  approximates  $(0, 0)$  as  $\lambda \uparrow \infty$  even in the regions where exactly one of the coefficients  $a$ , or  $d$ , vanishes, the other being positive; in spite of the cooperative structure of (1.1). Incidentally, this entails that the technical device developed by Molina-Meyer [13] to prove Theorem 1.1 under condition (1.14), based upon the construction of an appropriate supersolution, cannot be adapted to prove our Theorem 1.1. In this work, to accomplish that task, we are somehow obligated to adapt the scalar device introduced in [2], which goes back to Dancer and López-Gómez [5] and Kato [8, & IV.2.4].

If condition (1.6) is replaced by

$$\sigma_1[\mathcal{S}_0; \Omega_{0,2}] < \sigma_1[\mathcal{S}_0; \Omega_{0,1}],$$

then, instead of (1.11), we obtain that

$$\lim_{\lambda \uparrow \infty} \sigma_1[\mathcal{S}(\lambda a, \lambda d); \Omega] = \sigma_1[\mathcal{S}_0; \Omega_{0,2}],$$

and, instead of (1.13), we find that  $(\Phi_\omega, \Psi_\omega) = (0, 0)$  in  $\Omega \setminus \Omega_{0,2}$ , while  $(\Phi_\omega, \Psi_\omega)|_{\Omega_{0,2}}$  is the principal eigenfunction associated to  $\sigma_1[\mathcal{S}_0; \Omega_{0,2}]$ . When

$$\sigma_1[\mathcal{S}_0; \Omega_{0,1}] = \sigma_1[\mathcal{S}_0; \Omega_{0,2}],$$

then it remains an open problem to ascertain whether or not the limiting principal eigenfunction does concentrate either in  $\Omega_{0,1}$ , or in  $\Omega_{0,2}$ , or in both components simultaneously, however this is a rather classical problem going back to Simon [14, p. 110].

The distribution of this paper is as follows. Section 2 collects some known results that are going to be used in the proof of Theorem 1.1. Section 3 consists of the proof of Theorem 1.1. Finally, Section 4 shows that (1.6) holds if the Lebesgue measure of  $\Omega_{0,2}$  is sufficiently small.

Throughout this paper, for any  $D \subset \Omega$  and  $h \in \mathcal{C}(\bar{D})$ , it is said that  $h > 0$  if  $h \geq 0$  but  $h \neq 0$ . Similarly, given  $u, v \in \mathcal{C}(\bar{D})$ , it is said that  $(u, v) > (0, 0)$  if  $u \geq 0, v \geq 0$  and  $(u, v) \neq (0, 0)$ , and, given  $u_j, v_j \in \mathcal{C}(\bar{D})$ ,  $j \in \{1, 2\}$ , it is said that  $(u_1, v_1) > (u_2, v_2)$  if  $(u_1 - u_2, v_1 - v_2) > (0, 0)$ . Moreover, for any  $u, v \in C^1(\bar{D})$ , it is said that  $(u, v) \gg (0, 0)$  if  $u \gg 0$  and  $v \gg 0$ , and, as before, given  $u_j, v_j \in C^1(\bar{D})$ ,  $j \in \{1, 2\}$ , it is said that  $(u_1, v_1) \gg (u_2, v_2)$  if  $(u_1 - u_2, v_1 - v_2) \gg (0, 0)$ . Also, given two real Banach spaces  $U$  and  $V$  and a linear continuous operator  $T \in \mathcal{L}(U, V)$ , we shall denote by  $N[T]$  and  $R[T]$  the null space (kernel) and the rank (image) of  $T$ , respectively.

## 2. Maximum principle and principal eigenvalues

Throughout this paper we set

$$\mathcal{L} := \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} = \text{diag}\{L_1, L_2\}$$

and denote by  $\mathcal{C}_2$  the set of square matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathfrak{M}_2(\mathcal{C}^v(\bar{\Omega})) \quad (2.1)$$

of order 2 with entries in  $\mathcal{C}^v(\bar{\Omega})$  such that

$$a_{12}(x) > 0 \quad \text{and} \quad a_{21}(x) > 0 \quad \forall x \in \bar{\Omega}. \quad (2.2)$$

Then, according to [3,7,12,15], for every  $A \in \mathfrak{C}_2$  and any subdomain  $D$  of  $\Omega$  of class  $\mathcal{C}^{2,\nu}$ , there is a unique value of  $\tau$ , denoted by  $\sigma_1[\mathfrak{L} - A; D]$ , and called the principal eigenvalue of  $\mathfrak{L} - A$  in  $D$  (under homogeneous Dirichlet boundary conditions), for which

$$\begin{cases} (L_1 - a_{11})\varphi - a_{12}\psi = \tau\varphi \\ (L_2 - a_{22})\psi - a_{21}\varphi = \tau\psi \\ (\varphi, \psi) = (0, 0) \end{cases} \quad \begin{array}{l} \text{in } D, \\ \\ \text{on } \partial D, \end{array}$$

possesses a solution pair  $(\varphi, \psi)$  with  $\varphi > 0$  and  $\psi > 0$ . Moreover,  $\sigma_1[\mathfrak{L} - A; D]$  is algebraically *simple* and *dominant*, and the following characterization of the strong maximum principle, attributable to López-Gómez and Molina-Meyer [12], holds.

**Theorem 2.1.** *Suppose  $A \in \mathfrak{C}_2$  and  $D$  is an open subdomain of  $\Omega$  of class  $\mathcal{C}^{2,\nu}$ . Then, the following assertions are equivalent:*

- (a)  $\sigma_1[\mathfrak{L} - A, D] > 0$ .
- (b) *There exist  $h_1, h_2 \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$  such that  $(h_1, h_2) > (0, 0)$  in  $D$ ,*

$$(\mathfrak{L} - A) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } D,$$

*and either  $(h_1, h_2) > (0, 0)$  on  $\partial D$ , or else*

$$(\mathfrak{L} - A) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } D.$$

*Any of these pairs  $h := (h_1, h_2)$  is called a positive strict supersolution of  $\mathfrak{L} - A$  in  $D$ .*

- (c) *The operator  $\mathfrak{L} - A$  satisfies the strong maximum principle in  $D$ ; in the sense that  $f_1, f_2 \in \mathcal{C}^v(\bar{D})$ ,  $g_1, g_2 \in \mathcal{C}^{2,\nu}(\partial D)$ ,  $u, v \in \mathcal{C}^{2,\nu}(\bar{D})$ , and*

$$\begin{cases} (\mathfrak{L} - A) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ (u, v) = (g_1, g_2) \geq (0, 0) \end{cases} \quad \begin{array}{l} \text{in } D, \\ \text{on } \partial D, \end{array}$$

*with some inequality  $\geq$  strict, imply that  $u \gg 0$  and  $v \gg 0$  in  $D$ . In particular, any positive strict supersolution  $h$  of  $\mathfrak{L} - A$  in  $D$  satisfies  $h \gg 0$ .*

The following result collects some properties of principal eigenvalues going back to [7,12,15], thought it might be considered new as stated. Consequently, by the sake of completeness, we will give a short self-contained proof of it based upon Theorem 2.1.

**Proposition 2.2.** *The following assertions are true:*

- (a) *For every  $A, B \in \mathfrak{C}_2$  such that  $A \leq B$ ,  $A \neq B$ , the following estimate holds*

$$\sigma_1[\mathfrak{L} - A; \Omega] > \sigma_1[\mathfrak{L} - B; \Omega].$$

(b) Let  $a_{12}, a_{21} \in C^v(\bar{\Omega})$  satisfying (2.2). Then the mapping

$$(a_{11}, a_{22}) \in Z := C^v(\bar{\Omega}) \times C^v(\bar{\Omega}) \rightarrow \sigma_1[\mathcal{L} - A; \Omega] \in \mathbb{R},$$

where  $A$  is given by (2.1), is continuous. Actually, if  $(a_{11,n}, a_{22,n}) \in Z$ ,  $n \geq 1$ , is a sequence such that, for some  $a_{11}, a_{22} \in C^v(\bar{\Omega})$ ,

$$\lim_{n \rightarrow \infty} (a_{11,n}, a_{22,n}) = (a_{11}, a_{22}) \quad \text{uniformly in } \bar{\Omega},$$

then

$$\lim_{n \rightarrow \infty} \sigma_1[\mathcal{L} - A_n; \Omega] = \sigma_1[\mathcal{L} - A; \Omega], \quad (2.3)$$

where

$$A_n = \begin{pmatrix} a_{11,n} & a_{12} \\ a_{21} & a_{22,n} \end{pmatrix} \in \mathfrak{C}_2, \quad n \geq 1. \quad (2.4)$$

(c) If  $\Omega_0$  is a proper subdomain of  $\Omega$  of class  $C^{2,v}$ , then, for each  $A \in \mathfrak{C}_2$ ,

$$\sigma_1[\mathcal{L} - A; \Omega_0] > \sigma_1[\mathcal{L} - A; \Omega].$$

Property (a) goes back to [12, Theorem 3.2] and it is usually referred to as the *monotonicity property of the principal eigenvalue*  $\sigma[\mathcal{L} - A; \Omega]$  with respect to the potential matrix  $A$ . Property (b) establishes a *continuity of the principal eigenvalue with respect to the potential*, and property (c) establishes its *monotonicity with respect to the domain*.

**Proof.** (a) Let  $A, B \in \mathfrak{C}_2$ ,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

such that  $A \neq B$  and  $a_{ij} \leq b_{ij}$ ,  $i, j \in \{1, 2\}$ . Let  $(\varphi, \psi) \gg (0, 0)$  denote a principal eigenfunction associated to  $\sigma_1[\mathcal{L} - B; \Omega]$ . Then,  $(\varphi, \psi) = 0$  on  $\partial\Omega$ , and

$$(\mathcal{L} - A - \sigma_1[\mathcal{L} - B; \Omega] \operatorname{diag}\{1, 1\}) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = (B - A) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

because  $B > A$ ,  $\varphi \gg 0$  and  $\psi \gg 0$  in  $\Omega$ . Thus,  $(\varphi, \psi)$  provides us with a positive strict supersolution of

$$\tilde{\mathcal{L}} := \mathcal{L} - A - \sigma_1[\mathcal{L} - B; \Omega] \operatorname{diag}\{1, 1\}$$

in  $\Omega$ , and, therefore, according to Theorem 2.1,

$$0 < \sigma_1[\tilde{\mathcal{L}}; \Omega] = \sigma_1[\mathcal{L} - A; \Omega] - \sigma_1[\mathcal{L} - B; \Omega],$$

by uniqueness, which concludes the proof of part (a).

Now, we will prove part (b). Suppose the sequence  $A_n$ ,  $n \geq 1$ , defined through (2.4) satisfies all the requirements of part (b) and set (2.1). Then, for any  $\varepsilon > 0$ , there exists  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  such that, for every  $n \geq n_0$ ,

$$A - \varepsilon \operatorname{diag}\{1, 1\} \leq A_n \leq A + \varepsilon \operatorname{diag}\{1, 1\}.$$

Thus, by part (a), we find that, for any  $n \geq n_0$ ,

$$\sigma_1[\mathcal{L} - A - \varepsilon \operatorname{diag}\{1, 1\}; \Omega] \leq \sigma_1[\mathcal{L} - A_n; \Omega] \leq \sigma_1[\mathcal{L} - A + \varepsilon \operatorname{diag}\{1, 1\}; \Omega].$$

Consequently, by the uniqueness of the principal eigenvalue,

$$\sigma_1[\mathcal{L} - A; \Omega] - \varepsilon \leq \sigma_1[\mathcal{L} - A_n; \Omega] \leq \sigma_1[\mathcal{L} - A; \Omega] + \varepsilon, \quad n \geq n_0,$$

which concludes the proof.

To prove part (c), let  $\Omega_0$  be a proper subdomain of  $\Omega$  of class  $\mathcal{C}^{2,\nu}$  and  $A \in \mathfrak{C}_2$ . Let  $(\varphi, \psi) \gg (0, 0)$  be a principal eigenfunction associated to  $\sigma_1[\mathcal{L}; \Omega]$ . Then, since  $\partial\Omega_0 \cap \Omega \neq \emptyset$ , it is apparent that  $(\varphi, \psi) > (0, 0)$  on  $\partial\Omega_0$ . Moreover,

$$(\mathcal{L} - A - \sigma_1[\mathcal{L} - A; \Omega] \operatorname{diag}\{1, 1\}) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

in  $\Omega_0$  and, therefore,  $(\varphi, \psi)$  provides us with a positive strict supersolution of

$$\hat{\mathcal{L}} := \mathcal{L} - A - \sigma_1[\mathcal{L} - A; \Omega] \operatorname{diag}\{1, 1\}$$

in  $\Omega_0$ . Consequently, according to Theorem 2.1,

$$0 < \sigma_1[\hat{\mathcal{L}}; \Omega_0] = \sigma_1[\mathcal{L} - A; \Omega_0] - \sigma_1[\mathcal{L} - A; \Omega],$$

which ends the proof of the theorem.  $\square$

As an immediate consequence from Proposition 2.2, the following result holds.

**Corollary 2.3.** *Suppose (A1)–(A2). Then the real function  $\sigma(\lambda)$  defined through (1.9) is continuous, increasing and bounded above. Therefore, the limit*

$$\ell := \lim_{\lambda \rightarrow \infty} \sigma(\lambda) \tag{2.5}$$

*is well defined, i.e.,  $\ell \in \mathbb{R}$ . Moreover,*

$$\ell \leq \min\{\sigma_1[\mathcal{S}_0; \Omega_{0,1}], \sigma_1[\mathcal{S}_0; \Omega_{0,2}]\}.$$

**Proof.** The continuity is a consequence from Proposition 2.2(b). The monotonicity follows from Proposition 2.2(a). Now, pick  $k \in \{1, 2\}$ . Then, according to Proposition 2.2(c), we have that

$$\sigma(\lambda) = \sigma_1[\mathcal{S}(\lambda a, \lambda d); \Omega] < \sigma_1[\mathcal{S}(\lambda a, \lambda d); \Omega_{0,k}] = \sigma_1[\mathcal{S}_0; \Omega_{0,k}],$$

because  $a = d = 0$  in  $\Omega_{0,k}$ . This ends the proof.  $\square$



### 3. Proof of Theorem 1.1

It should be remembered that, for any  $\lambda \in \mathbb{R}$ , we have denoted by  $(\varphi_\lambda, \psi_\lambda)$  the unique principal eigenfunction associated to  $\sigma(\lambda) = \sigma_1[\mathcal{S}(\lambda a, \lambda d); \Omega]$  for which (1.10) holds. These eigenfunctions satisfy the following result.

**Lemma 3.1.** *There exists a constant  $C > 0$  such that*

$$\int_{\Omega} |\nabla \varphi_\lambda|^2 + \int_{\Omega} |\nabla \psi_\lambda|^2 \leq C \quad \text{for all } \lambda > 0.$$

**Proof.** By (A1), for each  $k \in \{1, 2\}$ , we have that

$$L_k = - \sum_{i,j=1}^N D_j(\alpha_{[ij,k]} D_i) + \sum_{i=1}^N \left( \alpha_{[i,k]} + \sum_{j=1}^N D_j \alpha_{[ij,k]} \right) D_i + \alpha_{[0,k]}.$$

Thus, setting

$$\mathfrak{a}_{[i,k]} := \alpha_{[i,k]} + \sum_{j=1}^N D_j \alpha_{[ij,k]}, \quad 1 \leq i \leq N, \quad k = 1, 2,$$

we have that  $\mathfrak{a}_{[i,k]} \in \mathcal{C}^{1,\nu}(\bar{\Omega})$ ,  $1 \leq i \leq N$ , and  $L_k$  can be expressed into the form

$$L_k = - \sum_{i,j=1}^N D_j(\alpha_{[ij,k]} D_i) + \sum_{i=1}^N \mathfrak{a}_{[i,k]} D_i + \alpha_{[0,k]}, \quad k = 1, 2.$$

Subsequently, we will make use of the fact that, for any  $\lambda \in \mathbb{R}$ ,

$$\begin{aligned} L_1 \varphi_\lambda &= -\lambda a \varphi_\lambda + b \psi_\lambda + \sigma(\lambda) \varphi_\lambda, \\ L_2 \psi_\lambda &= -\lambda d \psi_\lambda + c \varphi_\lambda + \sigma(\lambda) \psi_\lambda. \end{aligned} \tag{3.1}$$

Multiplying the first equation of (3.1) by  $\varphi_\lambda$ , the second one by  $\psi_\lambda$ , integrating the resulting identities in  $\Omega$ , and adding up the results, it becomes apparent that

$$\begin{aligned} & \sum_{i,j=1}^N \int_{\Omega} \alpha_{[ij,1]} D_i \varphi_\lambda D_j \varphi_\lambda + \sum_{i,j=1}^N \int_{\Omega} \alpha_{[ij,2]} D_i \psi_\lambda D_j \psi_\lambda + \sum_{i=1}^N \int_{\Omega} \mathfrak{a}_{[i,1]} \varphi_\lambda D_i \varphi_\lambda \\ & + \sum_{i=1}^N \int_{\Omega} \mathfrak{a}_{[i,2]} \psi_\lambda D_i \psi_\lambda + \int_{\Omega} \alpha_{[0,1]} \varphi_\lambda^2 + \int_{\Omega} \alpha_{[0,2]} \psi_\lambda^2 \\ & = -\lambda \left( \int_{\Omega} a \varphi_\lambda^2 + \int_{\Omega} d \psi_\lambda^2 \right) + \int_{\Omega} (b + c) \varphi_\lambda \psi_\lambda + \sigma(\lambda) \int_{\Omega} (\varphi_\lambda^2 + \psi_\lambda^2). \end{aligned}$$

Thus, due to (1.10) and Corollary 2.3, it follows from Hölder inequality that there exists a constant  $C_1 > 0$  such that, for every  $\lambda > 0$ ,

$$\begin{aligned} & \sum_{i,j=1}^N \int_{\Omega} \alpha_{[ij,1]} D_i \varphi_{\lambda} D_j \varphi_{\lambda} + \sum_{i,j=1}^N \int_{\Omega} \alpha_{[ij,2]} D_i \psi_{\lambda} D_j \psi_{\lambda} \\ & \leq C_1 - \sum_{i=1}^N \int_{\Omega} \alpha_{[i,1]} \varphi_{\lambda} D_i \varphi_{\lambda} - \sum_{i=1}^N \int_{\Omega} \alpha_{[i,2]} \psi_{\lambda} D_i \psi_{\lambda}. \end{aligned}$$

Consequently, by (1.3), we have that

$$\mu_1 \int_{\Omega} |\nabla \varphi_{\lambda}|^2 + \mu_2 \int_{\Omega} |\nabla \psi_{\lambda}|^2 \leq C_1 + \int_{\Omega} |\langle \alpha_1, \nabla \varphi_{\lambda} \rangle| \varphi_{\lambda} + \int_{\Omega} |\langle \alpha_2, \nabla \psi_{\lambda} \rangle| \psi_{\lambda},$$

where we have denoted

$$\alpha_k := (\alpha_{[1,k]}, \alpha_{[2,k]}, \dots, \alpha_{[N,k]}), \quad k = 1, 2, \quad (3.2)$$

and, therefore, there exists a constant  $C_2 > 0$ , such that, for any  $\lambda > 0$ , the following estimate holds

$$\mu_1 \int_{\Omega} |\nabla \varphi_{\lambda}|^2 + \mu_2 \int_{\Omega} |\nabla \psi_{\lambda}|^2 \leq C_1 + C_2 \left( \int_{\Omega} |\nabla \varphi_{\lambda}| \varphi_{\lambda} + \int_{\Omega} |\nabla \psi_{\lambda}| \psi_{\lambda} \right). \quad (3.3)$$

On the other hand, for each  $\eta > 0$ ,  $\lambda > 0$ , and  $u \in \{\varphi_{\lambda}, \psi_{\lambda}\}$ ,

$$u |\nabla u| = \eta u \frac{|\nabla u|}{\eta} \leq \frac{\eta^2}{2} u^2 + \frac{1}{2\eta^2} |\nabla u|^2, \quad (3.4)$$

and, hence, substituting these inequalities in (3.3) shows that

$$\mu_1 \int_{\Omega} |\nabla \varphi_{\lambda}|^2 + \mu_2 \int_{\Omega} |\nabla \psi_{\lambda}|^2 \leq C_1 + C_2 \frac{\eta^2}{2} + \frac{C_2}{2\eta^2} \int_{\Omega} (|\nabla \varphi_{\lambda}|^2 + |\nabla \psi_{\lambda}|^2). \quad (3.5)$$

By choosing any  $\eta$  such that

$$\min\{\mu_1, \mu_2\} > \frac{C_2}{2\eta^2},$$

it becomes apparent that (3.5) concludes the proof of the lemma.  $\square$

Let  $\{\lambda_n\}_{n \geq 1}$  be any increasing unbounded sequence, i.e., such that  $0 < \lambda_n < \lambda_m$  if  $n < m$  and

$$\lim_{n \rightarrow \infty} \lambda_n = \infty.$$

Then, for every  $n \geq 1$ ,

$$\int_{\Omega} (\varphi_{\lambda_n}^2 + \psi_{\lambda_n}^2) = 1, \quad (3.6)$$

and, owing to Lemma 3.1,  $\{(\varphi_{\lambda_n}, \psi_{\lambda_n})\}_{n \geq 1}$  is bounded in

$$X := H_0^1(\Omega) \times H_0^1(\Omega).$$

As the imbedding

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega)$$

is compact, there exists a subsequence of  $\{\lambda_n\}_{n \geq 1}$ , again labelled by  $n$ , and  $\varphi_{\omega}, \psi_{\omega} \in L^2(\Omega)$  such that

$$\lim_{n \rightarrow \infty} \|(\varphi_{\lambda_n}, \psi_{\lambda_n}) - (\varphi_{\omega}, \psi_{\omega})\|_Y = 0, \quad Y := L^2(\Omega) \times L^2(\Omega).$$

Next, we will prove that  $\{(\varphi_{\lambda_n}, \psi_{\lambda_n})\}_{n \geq 1}$  is actually a Cauchy sequence in  $X$ . This implies  $(\varphi_{\omega}, \psi_{\omega}) \in X$  and

$$\lim_{n \rightarrow \infty} \|(\varphi_{\lambda_n}, \psi_{\lambda_n}) - (\varphi_{\omega}, \psi_{\omega})\|_X = 0. \quad (3.7)$$

Fix  $n < m$ , and set

$$D_{n,m} := \mu_1 \int_{\Omega} |\nabla(\varphi_{\lambda_n} - \varphi_{\lambda_m})|^2 + \mu_2 \int_{\Omega} |\nabla(\psi_{\lambda_n} - \psi_{\lambda_m})|^2. \quad (3.8)$$

According to (1.3),

$$\begin{aligned} D_{n,m} &\leq \sum_{i,j=1}^N \int_{\Omega} \alpha_{[i,j,1]} D_i(\varphi_{\lambda_n} - \varphi_{\lambda_m}) D_j(\varphi_{\lambda_n} - \varphi_{\lambda_m}) \\ &\quad + \sum_{i,j=1}^N \int_{\Omega} \alpha_{[i,j,2]} D_i(\psi_{\lambda_n} - \psi_{\lambda_m}) D_j(\psi_{\lambda_n} - \psi_{\lambda_m}) \\ &= \sum_{i,j=1}^N \int_{\Omega} \alpha_{[i,j,1]} \{D_i \varphi_{\lambda_n} D_j \varphi_{\lambda_n} + D_i \varphi_{\lambda_m} D_j \varphi_{\lambda_m} - 2D_i \varphi_{\lambda_n} D_j \varphi_{\lambda_m}\} \\ &\quad + \sum_{i,j=1}^N \int_{\Omega} \alpha_{[i,j,2]} \{D_i \psi_{\lambda_n} D_j \psi_{\lambda_n} + D_i \psi_{\lambda_m} D_j \psi_{\lambda_m} - 2D_i \psi_{\lambda_n} D_j \psi_{\lambda_m}\}, \end{aligned}$$

and, hence, integrating by parts in  $\Omega$ , we are lead to

$$\begin{aligned}
D_{n,m} \leq & - \sum_{i,j=1}^N \int_{\Omega} \varphi_{\lambda_n} D_j(\alpha_{[ij,1]} D_i \varphi_{\lambda_n}) - \sum_{i,j=1}^N \int_{\Omega} \varphi_{\lambda_m} D_j(\alpha_{[ij,1]} D_i \varphi_{\lambda_m}) \\
& + 2 \sum_{i,j=1}^N \int_{\Omega} \varphi_{\lambda_m} D_j(\alpha_{[ij,1]} D_i \varphi_{\lambda_n}) - \sum_{i,j=1}^N \int_{\Omega} \psi_{\lambda_n} D_j(\alpha_{[ij,2]} D_i \psi_{\lambda_n}) \\
& - \sum_{i,j=1}^N \int_{\Omega} \psi_{\lambda_m} D_j(\alpha_{[ij,2]} D_i \psi_{\lambda_m}) + 2 \sum_{i,j=1}^N \int_{\Omega} \psi_{\lambda_m} D_j(\alpha_{[ij,2]} D_i \psi_{\lambda_n}),
\end{aligned}$$

because, for any  $h \geq 1$ ,

$$\varphi_{\lambda_h} = \psi_{\lambda_h} = 0 \quad \text{on } \partial\Omega.$$

Now, note that this inequality can be equivalently written in the form

$$\begin{aligned}
D_{n,m} \leq & \int_{\Omega} \left( L_1 \varphi_{\lambda_n} - \sum_{i=1}^N \mathfrak{a}_{[i,1]} D_i \varphi_{\lambda_n} - \alpha_{[0,1]} \varphi_{\lambda_n} \right) \varphi_{\lambda_n} \\
& + \int_{\Omega} \left( L_1 \varphi_{\lambda_m} - \sum_{i=1}^N \mathfrak{a}_{[i,1]} D_i \varphi_{\lambda_m} - \alpha_{[0,1]} \varphi_{\lambda_m} \right) \varphi_{\lambda_m} \\
& - 2 \int_{\Omega} \left( L_1 \varphi_{\lambda_n} - \sum_{i=1}^N \mathfrak{a}_{[i,1]} D_i \varphi_{\lambda_n} - \alpha_{[0,1]} \varphi_{\lambda_n} \right) \varphi_{\lambda_m} \\
& + \int_{\Omega} \left( L_2 \psi_{\lambda_n} - \sum_{i=1}^N \mathfrak{a}_{[i,2]} D_i \psi_{\lambda_n} - \alpha_{[0,2]} \psi_{\lambda_n} \right) \psi_{\lambda_n} \\
& + \int_{\Omega} \left( L_2 \psi_{\lambda_m} - \sum_{i=1}^N \mathfrak{a}_{[i,2]} D_i \psi_{\lambda_m} - \alpha_{[0,2]} \psi_{\lambda_m} \right) \psi_{\lambda_m} \\
& - 2 \int_{\Omega} \left( L_2 \psi_{\lambda_n} - \sum_{i=1}^N \mathfrak{a}_{[i,2]} D_i \psi_{\lambda_n} - \alpha_{[0,2]} \psi_{\lambda_n} \right) \psi_{\lambda_m}.
\end{aligned}$$

Consequently, by (3.1), we obtain that

$$\begin{aligned}
D_{n,m} \leq & \int_{\Omega} \left( \sigma(\lambda_n) \varphi_{\lambda_n} - \lambda_n a \varphi_{\lambda_n} - \sum_{i=1}^N \mathfrak{a}_{[i,1]} D_i \varphi_{\lambda_n} - \alpha_{[0,1]} \varphi_{\lambda_n} + b \psi_{\lambda_n} \right) \varphi_{\lambda_n} \\
& + \int_{\Omega} \left( \sigma(\lambda_m) \varphi_{\lambda_m} - \lambda_m a \varphi_{\lambda_m} - \sum_{i=1}^N \mathfrak{a}_{[i,1]} D_i \varphi_{\lambda_m} - \alpha_{[0,1]} \varphi_{\lambda_m} + b \psi_{\lambda_m} \right) \varphi_{\lambda_m}
\end{aligned}$$

$$\begin{aligned}
 & -2 \int_{\Omega} \left( \sigma(\lambda_n) \varphi_{\lambda_n} - \lambda_n a \varphi_{\lambda_n} - \sum_{i=1}^N \mathfrak{a}_{[i,1]} D_i \varphi_{\lambda_n} - \alpha_{[0,1]} \varphi_{\lambda_n} + b \psi_{\lambda_n} \right) \varphi_{\lambda_m} \\
 & + \int_{\Omega} \left( \sigma(\lambda_n) \psi_{\lambda_n} - \lambda_n d \psi_{\lambda_n} - \sum_{i=1}^N \mathfrak{a}_{[i,2]} D_i \psi_{\lambda_n} - \alpha_{[0,2]} \psi_{\lambda_n} + c \varphi_{\lambda_n} \right) \psi_{\lambda_n} \\
 & + \int_{\Omega} \left( \sigma(\lambda_m) \psi_{\lambda_m} - \lambda_m d \psi_{\lambda_m} - \sum_{i=1}^N \mathfrak{a}_{[i,2]} D_i \psi_{\lambda_m} - \alpha_{[0,2]} \psi_{\lambda_m} + c \varphi_{\lambda_m} \right) \psi_{\lambda_m} \\
 & - 2 \int_{\Omega} \left( \sigma(\lambda_n) \psi_{\lambda_n} - \lambda_n d \psi_{\lambda_n} - \sum_{i=1}^N \mathfrak{a}_{[i,2]} D_i \psi_{\lambda_n} - \alpha_{[0,2]} \psi_{\lambda_n} + c \varphi_{\lambda_n} \right) \psi_{\lambda_m}.
 \end{aligned}$$

Thus, rearranging terms, we are driven to the inequality

$$\begin{aligned}
 D_{n,m} & \leq \sigma(\lambda_n) \int_{\Omega} \varphi_{\lambda_n} (\varphi_{\lambda_n} - \varphi_{\lambda_m}) + \sigma(\lambda_n) \int_{\Omega} \psi_{\lambda_n} (\psi_{\lambda_n} - \psi_{\lambda_m}) \\
 & + \sigma(\lambda_m) \int_{\Omega} \varphi_{\lambda_m} (\varphi_{\lambda_m} - \varphi_{\lambda_n}) - \int_{\Omega} \alpha_{[0,1]} (\varphi_{\lambda_n} - \varphi_{\lambda_m})^2 \\
 & + \sigma(\lambda_m) \int_{\Omega} \psi_{\lambda_m} (\psi_{\lambda_m} - \psi_{\lambda_n}) - \int_{\Omega} \alpha_{[0,2]} (\psi_{\lambda_n} - \psi_{\lambda_m})^2 \\
 & + (\sigma(\lambda_m) - \sigma(\lambda_n)) \int_{\Omega} \varphi_{\lambda_n} \varphi_{\lambda_m} + (\sigma(\lambda_m) - \sigma(\lambda_n)) \int_{\Omega} \psi_{\lambda_n} \psi_{\lambda_m} \\
 & + \sum_{i=1}^N \int_{\Omega} \mathfrak{a}_{[i,1]} (\varphi_{\lambda_m} - \varphi_{\lambda_n}) D_i \varphi_{\lambda_n} + \sum_{i=1}^N \int_{\Omega} \mathfrak{a}_{[i,1]} \varphi_{\lambda_m} D_i (\varphi_{\lambda_n} - \varphi_{\lambda_m}) \\
 & + \sum_{i=1}^N \int_{\Omega} \mathfrak{a}_{[i,2]} (\psi_{\lambda_m} - \psi_{\lambda_n}) D_i \psi_{\lambda_n} + \sum_{i=1}^N \int_{\Omega} \mathfrak{a}_{[i,2]} \psi_{\lambda_m} D_i (\psi_{\lambda_n} - \psi_{\lambda_m}) \\
 & + \int_{\Omega} (\varphi_{\lambda_n} - \varphi_{\lambda_m}) (b \psi_{\lambda_n} - c \psi_{\lambda_m}) + \int_{\Omega} (\psi_{\lambda_n} - \psi_{\lambda_m}) (c \varphi_{\lambda_n} - b \varphi_{\lambda_m}) + R_{n,m},
 \end{aligned}$$

where we have denoted

$$\begin{aligned}
 R_{n,m} & := -\lambda_n \int_{\Omega} a \varphi_{\lambda_n}^2 - \lambda_m \int_{\Omega} a \varphi_{\lambda_m}^2 + 2\lambda_n \int_{\Omega} a \varphi_{\lambda_n} \varphi_{\lambda_m} \\
 & - \lambda_n \int_{\Omega} d \psi_{\lambda_n}^2 - \lambda_m \int_{\Omega} d \psi_{\lambda_m}^2 + 2\lambda_n \int_{\Omega} d \psi_{\lambda_n} \psi_{\lambda_m}.
 \end{aligned}$$

Obviously,

$$\begin{aligned} R_{n,m} = & -\lambda_n \int_{\Omega} a(\varphi_{\lambda_n} - \varphi_{\lambda_m})^2 + (\lambda_n - \lambda_m) \int_{\Omega} a\varphi_{\lambda_m}^2 \\ & - \lambda_n \int_{\Omega} d(\psi_{\lambda_n} - \psi_{\lambda_m})^2 + (\lambda_n - \lambda_m) \int_{\Omega} d\psi_{\lambda_m}^2 \leq 0, \end{aligned}$$

because, by construction,  $n < m$  implies  $\lambda_n < \lambda_m$ . Moreover, integrating by parts in  $\Omega$  shows that

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} \mathbf{a}_{[i,1]} \varphi_{\lambda_m} D_i(\varphi_{\lambda_n} - \varphi_{\lambda_m}) &= - \sum_{i=1}^N \int_{\Omega} (\varphi_{\lambda_n} - \varphi_{\lambda_m}) D_i(\mathbf{a}_{[i,1]} \varphi_{\lambda_m}), \\ \sum_{i=1}^N \int_{\Omega} \mathbf{a}_{[i,2]} \psi_{\lambda_m} D_i(\psi_{\lambda_n} - \psi_{\lambda_m}) &= - \sum_{i=1}^N \int_{\Omega} (\psi_{\lambda_n} - \psi_{\lambda_m}) D_i(\mathbf{a}_{[i,1]} \psi_{\lambda_m}). \end{aligned}$$

Therefore,

$$\begin{aligned} D_{n,m} \leq & \sigma(\lambda_n) \int_{\Omega} \varphi_{\lambda_n}(\varphi_{\lambda_n} - \varphi_{\lambda_m}) + \sigma(\lambda_n) \int_{\Omega} \psi_{\lambda_n}(\psi_{\lambda_n} - \psi_{\lambda_m}) \\ & + \sigma(\lambda_m) \int_{\Omega} \varphi_{\lambda_m}(\varphi_{\lambda_m} - \varphi_{\lambda_n}) - \int_{\Omega} \alpha_{[0,1]}(\varphi_{\lambda_n} - \varphi_{\lambda_m})^2 \\ & + \sigma(\lambda_m) \int_{\Omega} \psi_{\lambda_m}(\psi_{\lambda_m} - \psi_{\lambda_n}) - \int_{\Omega} \alpha_{[0,2]}(\psi_{\lambda_n} - \psi_{\lambda_m})^2 \\ & + (\sigma(\lambda_m) - \sigma(\lambda_n)) \int_{\Omega} \varphi_{\lambda_n} \varphi_{\lambda_m} + (\sigma(\lambda_m) - \sigma(\lambda_n)) \int_{\Omega} \psi_{\lambda_n} \psi_{\lambda_m} \\ & + \sum_{i=1}^N \int_{\Omega} \mathbf{a}_{[i,1]}(\varphi_{\lambda_m} - \varphi_{\lambda_n}) D_i \varphi_{\lambda_n} - \sum_{i=1}^N \int_{\Omega} (\varphi_{\lambda_n} - \varphi_{\lambda_m}) D_i(\mathbf{a}_{[i,1]} \varphi_{\lambda_m}) \\ & + \sum_{i=1}^N \int_{\Omega} \mathbf{a}_{[i,2]}(\psi_{\lambda_m} - \psi_{\lambda_n}) D_i \psi_{\lambda_n} - \sum_{i=1}^N \int_{\Omega} (\psi_{\lambda_n} - \psi_{\lambda_m}) D_i(\mathbf{a}_{[i,2]} \psi_{\lambda_m}) \\ & + \int_{\Omega} (\varphi_{\lambda_n} - \varphi_{\lambda_m})(b\psi_{\lambda_n} - c\psi_{\lambda_m}) + \int_{\Omega} (\psi_{\lambda_n} - \psi_{\lambda_m})(c\varphi_{\lambda_n} - b\varphi_{\lambda_m}). \end{aligned} \quad (3.9)$$

According to (3.6), it follows from Hölder inequality that

$$\left| \int_{\Omega} \varphi_{\lambda_n}(\varphi_{\lambda_n} - \varphi_{\lambda_m}) + \int_{\Omega} \psi_{\lambda_n}(\psi_{\lambda_n} - \psi_{\lambda_m}) \right| \leq \|(\varphi_{\lambda_n}, \psi_{\lambda_n}) - (\varphi_{\lambda_m}, \psi_{\lambda_m})\|_Y,$$

$$\left| \int_{\Omega} \varphi_{\lambda_m} (\varphi_{\lambda_n} - \varphi_{\lambda_n}) + \int_{\Omega} \psi_{\lambda_m} (\psi_{\lambda_n} - \psi_{\lambda_n}) \right| \leq \|(\varphi_{\lambda_m}, \psi_{\lambda_m}) - (\varphi_{\lambda_n}, \psi_{\lambda_n})\|_Y,$$

where we are denoting  $Y := L^2(\Omega) \times L^2(\Omega)$ . Moreover,

$$\begin{aligned} - \int_{\Omega} \alpha_{[0,1]} (\varphi_{\lambda_n} - \varphi_{\lambda_m})^2 &\leq - \inf_{\Omega} \alpha_{[0,1]} \|\varphi_{\lambda_n} - \varphi_{\lambda_m}\|_{L^2(\Omega)}^2, \\ - \int_{\Omega} \alpha_{[0,2]} (\psi_{\lambda_n} - \psi_{\lambda_m})^2 &\leq - \inf_{\Omega} \alpha_{[0,2]} \|\psi_{\lambda_n} - \psi_{\lambda_m}\|_{L^2(\Omega)}^2, \\ \left| [\sigma(\lambda_m) - \sigma(\lambda_n)] \int_{\Omega} (\varphi_{\lambda_n} \varphi_{\lambda_m} + \psi_{\lambda_n} \psi_{\lambda_m}) \right| &\leq 2 |\sigma(\lambda_m) - \sigma(\lambda_n)|, \end{aligned}$$

and, making use of (3.2), we find from Hölder inequality and Lemma 3.1 that

$$\begin{aligned} \left| \sum_{i=1}^N \int_{\Omega} \mathbf{a}_{[i,1]} (\varphi_{\lambda_m} - \varphi_{\lambda_n}) D_i \varphi_{\lambda_n} \right| &= \left| \int_{\Omega} \langle \mathbf{a}_1, \nabla \varphi_{\lambda_n} \rangle (\varphi_{\lambda_m} - \varphi_{\lambda_n}) \right| \\ &\leq C_1 \int_{\Omega} |\nabla \varphi_{\lambda_n}| |\varphi_{\lambda_m} - \varphi_{\lambda_n}| \leq C_2 \|\varphi_{\lambda_m} - \varphi_{\lambda_n}\|_{L^2(\Omega)}, \end{aligned}$$

for some positive constants  $C_1 > 0$  and  $C_2 > 0$ , whose explicit knowledge is not important here. Similarly, there exist constants  $C_3 > 0, \dots, C_7 > 0$ , such that

$$\begin{aligned} \left| \sum_{i=1}^N \int_{\Omega} \mathbf{a}_{[i,2]} (\psi_{\lambda_m} - \psi_{\lambda_n}) D_i \psi_{\lambda_n} \right| &\leq C_3 \|\psi_{\lambda_m} - \psi_{\lambda_n}\|_{L^2(\Omega)}, \\ \left| \sum_{i=1}^N \int_{\Omega} (\varphi_{\lambda_n} - \varphi_{\lambda_m}) D_i (\mathbf{a}_{[i,1]} \varphi_{\lambda_m}) \right| &\leq C_4 \|\varphi_{\lambda_m} - \varphi_{\lambda_n}\|_{L^2(\Omega)}, \\ \left| \sum_{i=1}^N \int_{\Omega} (\psi_{\lambda_n} - \psi_{\lambda_m}) D_i (\mathbf{a}_{[i,2]} \psi_{\lambda_m}) \right| &\leq C_5 \|\psi_{\lambda_m} - \psi_{\lambda_n}\|_{L^2(\Omega)}, \\ \left| \int_{\Omega} (\varphi_{\lambda_n} - \varphi_{\lambda_m}) (b \psi_{\lambda_n} - c \psi_{\lambda_m}) \right| &\leq C_6 \|\varphi_{\lambda_m} - \varphi_{\lambda_n}\|_{L^2(\Omega)}, \\ \left| \int_{\Omega} (\psi_{\lambda_n} - \psi_{\lambda_m}) (c \varphi_{\lambda_n} - b \varphi_{\lambda_m}) \right| &\leq C_7 \|\psi_{\lambda_n} - \psi_{\lambda_m}\|_{L^2(\Omega)}. \end{aligned}$$

Thus, substituting these inequalities into (3.9) and using (2.5), we find that there exists  $C_8 > 0$  such that

$$D_{n,m} \leq C_8 (\|\varphi_{\lambda_m} - \varphi_{\lambda_n}\|_{L^2(\Omega)} + \|\psi_{\lambda_m} - \psi_{\lambda_n}\|_{L^2(\Omega)}) + 2 |\sigma(\lambda_m) - \sigma(\lambda_n)|. \quad (3.10)$$

Consequently, from (3.8) and (3.10) it becomes apparent that there exists a constant  $C = C(\eta) > 0$  such that

$$\begin{aligned} \int_{\Omega} |\nabla(\varphi_{\lambda_n} - \varphi_{\lambda_m})|^2 + \int_{\Omega} |\nabla(\psi_{\lambda_n} - \psi_{\lambda_m})|^2 &\leq C |\sigma(\lambda_m) - \sigma(\lambda_n)| \\ &+ C (\|\varphi_{\lambda_m} - \varphi_{\lambda_n}\|_{L^2(\Omega)} + \|\psi_{\lambda_m} - \psi_{\lambda_n}\|_{L^2(\Omega)}). \end{aligned}$$

This shows that indeed  $\{(\varphi_{\lambda_n}, \psi_{\lambda_n})\}_{n \geq 1}$  is a Cauchy sequence in  $X$ . Therefore, (3.7) holds. Note that, taking limits, we also find that

$$(\varphi_{\omega}, \psi_{\omega}) \geq (0, 0) \quad \text{and} \quad \int_{\Omega} (\varphi_{\omega}^2 + \psi_{\omega}^2) = 1. \quad (3.11)$$

Next, we will show that

$$(\varphi_{\omega}, \psi_{\omega}) = (0, 0) \quad \text{in } \Omega_+ = \Omega_+^a \cup \Omega_+^d. \quad (3.12)$$

According to the proof of Lemma 3.1, we already know that, for every  $n \geq 1$ ,

$$\begin{aligned} &\sum_{i,j=1}^N \int_{\Omega} \alpha_{[ij,1]} D_i \varphi_{\lambda_n} D_j \varphi_{\lambda_n} + \sum_{i,j=1}^N \int_{\Omega} \alpha_{[ij,2]} D_i \psi_{\lambda_n} D_j \psi_{\lambda_n} + \sum_{i=1}^N \int_{\Omega} \alpha_{[i,1]} \varphi_{\lambda_n} D_i \varphi_{\lambda_n} \\ &\quad + \sum_{i=1}^N \int_{\Omega} \alpha_{[i,2]} \psi_{\lambda_n} D_i \psi_{\lambda_n} + \int_{\Omega} \alpha_{[0,1]} \varphi_{\lambda_n}^2 + \int_{\Omega} \alpha_{[0,2]} \psi_{\lambda_n}^2 \\ &= -\lambda_n \left( \int_{\Omega} a \varphi_{\lambda_n}^2 + \int_{\Omega} d \psi_{\lambda_n}^2 \right) + \int_{\Omega} (b+c) \varphi_{\lambda_n} \psi_{\lambda_n} + \sigma(\lambda_n). \end{aligned}$$

According to (3.7) and Corollary 2.3, taking limits as  $n \rightarrow \infty$  in this identity, the theorem of dominated convergence of Lebesgue makes sure that

$$\lim_{n \rightarrow \infty} \int_{\Omega} (a \varphi_{\lambda_n}^2 + d \psi_{\lambda_n}^2) = 0,$$

for as  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ . On the other hand, by Hölder inequality, it follows from (3.6) and (3.11) that

$$\begin{aligned} \left| \int_{\Omega} a \varphi_{\lambda_n}^2 - \int_{\Omega} a \varphi_{\omega}^2 \right| &\leq \int_{\Omega} a(\varphi_{\lambda_n} + \varphi_{\omega}) |\varphi_{\lambda_n} - \varphi_{\omega}| \\ &\leq \max_{\Omega} a \cdot \|\varphi_{\lambda_n} - \varphi_{\omega}\|_{L^2(\Omega)} \left( \int_{\Omega} \varphi_{\lambda_n}^2 + \int_{\Omega} \varphi_{\omega}^2 + 2 \int_{\Omega} \varphi_{\lambda_n} \varphi_{\omega} \right)^{1/2} \\ &\leq 2 \max_{\Omega} a \cdot \|\varphi_{\lambda_n} - \varphi_{\omega}\|_{L^2(\Omega)}. \end{aligned}$$



Similarly,

$$\left| \int_{\Omega} d\psi_{\lambda_n}^2 - \int_{\Omega} d\psi_{\omega}^2 \right| \leq 2 \max_{\bar{\Omega}} d \cdot \|\psi_{\lambda_n} - \psi_{\omega}\|_{L^2(\Omega)}.$$

Therefore,

$$\int_{\Omega} (a\varphi_{\omega}^2 + d\psi_{\omega}^2) = 0,$$

which concludes the proof of (3.12). Next, we show that (3.12) implies that

$$(\varphi_{\omega}, \psi_{\omega})|_{\Omega_{0,k}} \in H_0^1(\Omega_{0,k}) \times H_0^1(\Omega_{0,k}), \quad k \in \{1, 2\}. \quad (3.13)$$

Indeed, for each  $k \in \{1, 2\}$  and sufficiently small  $\delta > 0$ , consider the open set

$$\Omega_{\delta,k} := \{x \in \Omega : \text{dist}(x, \partial\Omega_{0,k}) < \delta\}.$$

According to (3.12),

$$(\varphi_{\omega}, \psi_{\omega})|_{\Omega_{\delta,k}} \in H_0^1(\Omega_{\delta,k}) \times H_0^1(\Omega_{\delta,k}), \quad k \in \{1, 2\},$$

and, hence, there exists  $\delta_0 > 0$  such that

$$(\varphi_{\omega}, \psi_{\omega}) \in \bigcap_{0 < \delta < \delta_0} (H_0^1(\Omega_{\delta,k}) \times H_0^1(\Omega_{\delta,k})), \quad k \in \{1, 2\}.$$

On the other hand, since  $\Omega_{0,1}$  and  $\Omega_{0,2}$  are smooth subdomains of  $\Omega$ , they are stable in the sense of Babuska and Výměny [4] (cf. López-Gómez [11]) and, therefore,

$$H_0^1(\Omega_{0,k}) \times H_0^1(\Omega_{0,k}) = \bigcap_{0 < \delta < \delta_0} (H_0^1(\Omega_{\delta,k}) \times H_0^1(\Omega_{\delta,k})), \quad k \in \{1, 2\},$$

which concludes the proof of (3.13).

Subsequently, we pick  $k \in \{1, 2\}$  and a test function

$$(\xi_1, \xi_2) \in C_0^{\infty}(\Omega_{0,k}) \times C_0^{\infty}(\Omega_{0,k}).$$

Particularizing (3.1) at  $\lambda = \lambda_n$ ,  $n \geq 1$ , multiplying the first equation by  $\xi_1$ , the second one by  $\xi_2$ , and integrating the resulting identities in  $\Omega$  gives rise to

$$\begin{aligned} & \sum_{i,j=1}^N \int_{\Omega_{0,k}} \alpha_{[ij,1]} D_i \xi_1 D_j \varphi_{\lambda_n} + \sum_{i=1}^N \int_{\Omega_{0,k}} \alpha_{[i,1]} \xi_1 D_i \varphi_{\lambda_n} + \int_{\Omega_{0,k}} \alpha_{[0,1]} \varphi_{\lambda_n} \xi_1 \\ &= \int_{\Omega_{0,k}} b \psi_{\lambda_n} \xi_1 + \sigma(\lambda_n) \int_{\Omega_{0,k}} \xi_1 \varphi_{\lambda_n}, \end{aligned}$$

$$\begin{aligned}
& \sum_{i,j=1}^N \int_{\Omega_{0,k}} \alpha_{[ij,2]} D_i \xi_2 D_j \psi_{\lambda_n} + \sum_{i=1}^N \int_{\Omega_{0,k}} \alpha_{[i,2]} \xi_2 D_i \psi_{\lambda_n} + \int_{\Omega_{0,k}} \alpha_{[0,2]} \psi_{\lambda_n} \xi_2 \\
&= \int_{\Omega_{0,k}} c \varphi_{\lambda_n} \xi_2 + \sigma(\lambda_n) \int_{\Omega_{0,k}} \xi_2 \psi_{\lambda_n}.
\end{aligned}$$

Consequently, passing to the limit as  $n \rightarrow \infty$ , it is apparent that  $(\varphi_\omega, \psi_\omega)|_{\Omega_{0,k}}$  provides us with a weak solution of the problem

$$S_0 \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \ell \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \quad \text{in } \Omega_{0,k}, \quad (\varphi, \psi) = (0, 0) \quad \text{on } \partial\Omega_{0,k}, \quad (3.14)$$

where  $\ell$  is given through (2.5) and

$$S_0 := \begin{pmatrix} L_1 & -b \\ -c & L_2 \end{pmatrix}.$$

By elliptic regularity, it is easy to realize that

$$(\varphi_\omega, \psi_\omega)|_{\Omega_{0,k}} \in \mathcal{C}^{2,\nu}(\bar{\Omega}_{0,k}) \times \mathcal{C}^{2,\nu}(\bar{\Omega}_{0,k}), \quad k \in \{1, 2\},$$

is a classical solution of (3.14). Note that, owing to (3.11),

$$(\varphi_\omega, \psi_\omega) > (0, 0) \quad \text{in } \Omega_{0,1} \cup \Omega_{0,2}.$$

Moreover, for each  $k \in \{1, 2\}$ , either  $\varphi_\omega = \psi_\omega = 0$  in  $\Omega_{0,k}$ , or else  $\varphi_\omega > 0$  and  $\psi_\omega > 0$  in  $\Omega_{0,k}$ . Indeed, if, for instance,  $\varphi_\omega = 0$  in  $\Omega_{0,k}$ , then  $b\psi_\omega = 0$  in  $\Omega_{0,k}$ , which implies  $\psi_\omega = 0$ . Similarly,  $\varphi_\omega = 0$  in  $\Omega_{0,k}$  if  $\psi_\omega = 0$  therein. Therefore, according to (1.6), we conclude from Theorem 2.1 and Corollary 2.3 that

$$(\varphi_\omega, \psi_\omega) = (0, 0) \quad \text{in } \Omega_{0,2}, \quad \ell = \sigma_1[S_0; \Omega_{0,1}],$$

and

$$(\varphi_\omega, \psi_\omega) \gg (0, 0) \quad \text{in } \Omega_{0,1}.$$

This concludes the proof of Theorem 1.1.

#### 4. Lower estimates of $\sigma_1[S_0; \Omega]$ as $|\Omega| \downarrow 0$

Throughout this section, for a given subdomain  $D$  of  $\Omega$ ,  $|D|$  stands for the Lebesgue measure of  $D$ . The following result guarantees that (1.6) holds if  $|\Omega_{0,2}|$  is sufficiently small; it is the main result of this section; it extends to [11, Theorem 5.1].

**Theorem 4.1.** *Let  $S$  denote the operator matrix defined in (1.5), and  $S_0 := S(0, 0)$ . Then, for sufficiently small  $|\Omega|$ ,*

$$\sigma_1[S_0; \Omega] \geq \mu \Sigma |B_1|^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} - \alpha \sqrt{2\Sigma} |B_1|^{\frac{1}{N}} |\Omega|^{-\frac{1}{N}} - \beta, \quad (4.1)$$

where  $B_1$  is the unit ball of  $\mathbb{R}^N$ ,

$$\Sigma := \sigma_1[-\Delta; B_1], \quad \mu := \min\{\mu_1, \mu_2\}/2, \quad \alpha := \max\{\|\mathbf{a}_1\|_\infty, \|\mathbf{a}_2\|_\infty\}/2,$$

and

$$\beta = \frac{1}{2} \left( \left| \min_{\bar{\Omega}} \alpha_{[0,1]} \right| + \left| \min_{\bar{\Omega}} \alpha_{[0,2]} \right| + \max_{\bar{\Omega}} b + \max_{\bar{\Omega}} c \right).$$

In particular,

$$\lim_{|\Omega| \rightarrow 0} \sigma_1[\mathcal{S}_0; \Omega] = \infty.$$

**Proof.** Let  $(\varphi, \psi)$  denote the principal eigenfunction associated to  $\sigma_1[\mathcal{S}_0; \Omega]$ , normalized so that

$$\int_{\Omega} (\varphi^2 + \psi^2) = 1. \quad (4.2)$$

Then,

$$L_1\varphi = b\psi + \sigma_1[\mathcal{S}_0; \Omega]\varphi$$

and, hence,

$$\begin{aligned} \int_{\Omega} \varphi L_1\varphi &= \int_{\Omega} b\psi\varphi + \sigma_1[\mathcal{S}_0; \Omega] \int_{\Omega} \varphi^2 \\ &\leq \max_{\bar{\Omega}} b \left( \int_{\Omega} \psi^2 \right)^{1/2} \left( \int_{\Omega} \varphi^2 \right)^{1/2} + \sigma_1[\mathcal{S}_0; \Omega] \int_{\Omega} \varphi^2 \\ &\leq \max_{\bar{\Omega}} b + \sigma_1[\mathcal{S}_0; \Omega]. \end{aligned}$$

Consequently,

$$\begin{aligned} \sigma_1[\mathcal{S}_0; \Omega] &\geq \int_{\Omega} \varphi L_1\varphi - \max_{\bar{\Omega}} b \\ &= \sum_{i,j=1}^N \int_{\Omega} \alpha_{[ij,1]} D_i\varphi D_j\varphi + \sum_{i=1}^N \int_{\Omega} \mathbf{a}_{[i,1]}\varphi D_i\varphi + \int_{\Omega} \alpha_{[0,1]}\varphi^2 - \max_{\bar{\Omega}} b \\ &\geq \mu_1 \int_{\Omega} |\nabla\varphi|^2 + \int_{\Omega} \varphi \langle \mathbf{a}_1, \nabla\varphi \rangle - \left| \min_{\bar{\Omega}} \alpha_{[0,1]} \right| - \max_{\bar{\Omega}} b, \end{aligned}$$

where  $\mathbf{a}_1$  is given through (3.2). Moreover,

$$\left| \int_{\Omega} \varphi \langle \mathbf{a}_1, \nabla \varphi \rangle \right| \leq \|\mathbf{a}_1\|_{\infty} \int_{\Omega} \varphi |\nabla \varphi| \leq \|\mathbf{a}_1\|_{\infty} \|\nabla \varphi\|_{L^2(\Omega)},$$

because  $\|\varphi\|_{L^2(\Omega)} \leq 1$ . Thus,

$$\sigma_1[\mathcal{S}_0; \Omega] \geq \mu_1 \int_{\Omega} |\nabla \varphi|^2 - \|\mathbf{a}_1\|_{\infty} \left( \int_{\Omega} |\nabla \varphi|^2 \right)^{1/2} - \left| \min_{\Omega} \alpha_{[0,1]} \right| - \max_{\Omega} b.$$

Similarly, it follows from

$$L_2 \psi = c\varphi + \sigma_1[\mathcal{S}_0; \Omega] \psi$$

that

$$\sigma_1[\mathcal{S}_0; \Omega] \geq \mu_2 \int_{\Omega} |\nabla \psi|^2 - \|\mathbf{a}_2\|_{\infty} \left( \int_{\Omega} |\nabla \psi|^2 \right)^{1/2} - \left| \min_{\Omega} \alpha_{[0,2]} \right| - \max_{\Omega} c.$$

Consequently, adding up these two inequalities shows that

$$\sigma_1[\mathcal{S}_0; \Omega] \geq \mu \left( \int_{\Omega} |\nabla \varphi|^2 + \int_{\Omega} |\nabla \psi|^2 \right) - \alpha \left[ \left( \int_{\Omega} |\nabla \varphi|^2 \right)^{\frac{1}{2}} + \left( \int_{\Omega} |\nabla \psi|^2 \right)^{\frac{1}{2}} \right] - \beta.$$

On the other hand, the following estimate

$$\sqrt{x^2 + y^2} \leq x + y \leq \sqrt{2} \sqrt{x^2 + y^2}, \quad x \geq 0, \quad y \geq 0,$$

implies that

$$\left( \int_{\Omega} |\nabla \varphi|^2 \right)^{\frac{1}{2}} + \left( \int_{\Omega} |\nabla \psi|^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \left( \int_{\Omega} |\nabla \varphi|^2 + \int_{\Omega} |\nabla \psi|^2 \right)^{\frac{1}{2}}$$

and, hence,

$$\begin{aligned} \sigma_1[\mathcal{S}_0; \Omega] &\geq \mu \left( \int_{\Omega} |\nabla \varphi|^2 + \int_{\Omega} |\nabla \psi|^2 \right) - \alpha \sqrt{2} \left( \int_{\Omega} |\nabla \varphi|^2 + \int_{\Omega} |\nabla \psi|^2 \right)^{\frac{1}{2}} - \beta \\ &= \left( \int_{\Omega} |\nabla \varphi|^2 + \int_{\Omega} |\nabla \psi|^2 \right)^{\frac{1}{2}} \left[ \mu \left( \int_{\Omega} |\nabla \varphi|^2 + \int_{\Omega} |\nabla \psi|^2 \right)^{\frac{1}{2}} - \alpha \sqrt{2} \right] - \beta. \end{aligned}$$

According to (4.2), by the variational characterization of  $\sigma_1[-\Delta; \Omega]$ , we have that

$$\int_{\Omega} |\nabla \varphi|^2 + \int_{\Omega} |\nabla \psi|^2 \geq \sigma_1[-\Delta; \Omega] \left( \int_{\Omega} \varphi^2 + \int_{\Omega} \psi^2 \right) = \sigma_1[-\Delta; \Omega].$$

Moreover, by a well-known inequality of Faber [6] and Krahn [9] (see [11, p. 280]), it is well known that

$$\sigma_1[-\Delta; \Omega] \geq \Sigma |B_1|^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}.$$

Therefore, for sufficiently small  $|\Omega|$ , (4.1) holds. This concludes the proof.  $\square$

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